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Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method ☆

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Abstract

This study obtains two semilocal convergence results for the well-known Chebyshev method, which is a third-order iterative process. The hypotheses required are modifications to the normal Kantorovich ones. The results obtained are applied to the reduction of nonlinear integral equations of the Fredholm type and first kind.

MSC: 47H17; 65J15

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1. Introduction

Chebyshev's method [2,4] is one of the best-known third-order iterative processes for the resolution of nonlinear equations of the following form:

$$F(x) = 0.$$

The algorithm which defines it is one of the simplest which can be obtained for third-order point-to-point iterative processes, as can be deduced from the characterisation given by Gander [6] for these iterative processes. For example, if X and Y are Banach spaces and $F : \Omega \subseteq X \to Y$ is a nonlinear twice-differentiable Frèchet operator defined on a convex, nonempty domain Ω, the algorithm which

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defines Chebyshev's method is given by

$$x_{n+1} = x_n - [I + \tfrac{1}{2}L_F(x_n)]F'(x_n)^{-1}F(x_n),$$

where, for $x \in X$, $L_F(x)$ is the linear operator defined as follows:

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x).$$

The properties of this operator can be seen in [7].

In general, the convergence of third-order methods has been proved in bounded conditions for the operator's third derivative [1] or else with a Lipschitz-type condition for the second derivative together with the requirement that this be bounded [2].

This study has its origins in [8], in which a semilocal convergence result for the Chebyshev method was proved when applied to continuous (K, p)-Hölder twice-differentiable Frèchet operators; this supposed a milder convergence conditions for this method. In the study described here, we further smoothen the conditions imposed on operator F. In Section 3 we prove a semilocal convergence result which differs from that obtained in [8] because it does not need even the existence of the operator's Frèchet third derivative. The only requirement is that the operator's Frèchet second derivative be bounded and that a Lipschitz-type condition be verified. In Section 4 we again improve the result obtained, eliminating the condition that the operator's Frèchet second derivative be bounded. These major improvements to the semilocal convergence results for Chebyshev's method are principally due to the technique used. The most commonly used technique is majorant sequences [1,2]; in this study, we construct some real successions and so obtain certain recurrence relations which allow us to check the convergence of the Chebyshev method while reducing to the minimum possible the conditions required of the operator F. Furthermore, the error bounds which we obtain for the application of the Chebyshev method are sufficiently competitive, a situation which we justify by example in Section 3.

The results obtained are then applied to the resolution of particular nonlinear integral equations of the Fredholm type and first kind.

2. Preliminaries

If we define $\Gamma_n = F'(x_n)^{-1}$, we can write the Chebyshev method in the form

$$\begin{aligned} y_n &= x_n - \Gamma_n F(x_n), \\ x_{n+1} &= y_n + \tfrac{1}{2}L_F(x_n)(y_n - x_n). \end{aligned} \tag{1}$$

Let us assume that $F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists for some $x_0 \in \Omega$, where $\mathscr{L}(Y,X)$ is the set of bounded linear operators from Y into X. Moreover we suppose that

(c_1) $\|\Gamma_0\| \leqslant \beta$,

(c_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta$.

(c_3) $\|F''(x)\| \leqslant M$, $x \in \Omega$

(c_4) $\|F''(x) - F''(y)\| \leqslant K\|x - y\|^p$, $x, y \in \Omega$, $K > 0$, $p \in [0, 1]$, We denote

$$a_0 = M\beta\eta, \quad b_0 = K\beta\eta^{p+1}, \tag{2}$$

$$f(x) = \frac{2}{2 - 2x - x^2}, \tag{3}$$

$$g(x, y) = \frac{x^2}{2} + \frac{x^3}{8} + \frac{y}{(p+1)(p+2)}. \tag{4}$$

and define the sequences

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n) \quad \text{and} \quad b_{n+1} = b_n f(a_n)^{p+2} g(a_n, b_n)^{p+1}.$$

Firstly, a technical lemma is provided whose proof is trivial.

Lemma 2.1. *Let f and g be two real functions given in* (3) *and* (4) *respectively, and let $p \in [0, 1]$. Then*

(i) *f is increasing and $f(x) > 1$ for $x \in (0, \frac{1}{2})$,*
(ii) *For a fixed $x \in (0, \frac{1}{2})$, $g(x, y)$ increases as a function of y, and for a fixed $y > 0$, $g(x, y)$ increases in $(0, \frac{1}{2})$,*
(iii) *$f(\gamma x) < f(x)$ and $g(\gamma x, \gamma^{p+1} y) \leqslant \gamma^{p+1} g(x, y)$ for $x \in (0, \frac{1}{2})$, $y > 0$ and $\gamma \in (0, 1)$.*

Some properties for the sequence $\{a_n\}$ and $\{b_n\}$ are now provided.

Lemma 2.2. *Let $0 < a_0 < \frac{1}{2}$ and $f(a_0)^2 g(a_0, b_0) < 1$. Then the sequences $\{a_n\}$ and $\{b_n\}$ are decreasing.*

Proof. From the hypothesis, we deduce that $0 < a_1 < a_0$ and $0 < b_1 < b_0$, since $f(x) > 1$ in $(0, \frac{1}{2})$. Now we suppose that $0 < a_k < a_{k-1} < \cdots < a_1 < a_0 < \frac{1}{2}$ and $0 < b_k < b_{k-1} < \cdots < b_1 < b_0$. Then, $0 < a_{k+1} < a_k$ if and only if $f(a_k)^2 g(a_k, b_k) < 1$.

Notice that $f(a_k) < f(a_0)$ and $g(a_k, b_k) < g(a_0, b_k) < g(a_0, b_0)$. Consequently, $f(a_k)^2 g(a_k, b_k) < 1$.

Now, the fact of demonstrating $b_{k+1} \leqslant b_k$ is equivalent to proving that $f(a_k)^{p+2} g(a_k, b_k)^{p+1} < 1$. Taking into account $b_{k+1} > 0$ and following the previous reasoning, the result also holds. □

In the following lemma, whose proof is obvious, we give sufficient conditions so that the real sequences $\{a_n\}$ and $\{b_n\}$ are decreasing.

Lemma 2.3. *If $0 < a_0 < \frac{1}{2}$ and $b_0 < ((p+1)(p+2)/8)(1 - 2a_0)(a_0 + 2)(4 - 2a_0 - a_0^2)$, then $f(a_0)^2 g(a_0, b_0) < 1$.*

Lemma 2.4. *Let us suppose that the hypotheses of Lemma 2.3 are satisfied and define $\gamma = a_1/a_0$. Then*

(i) *$\gamma = f(a_0)^2 g(a_0, b_0) \in (0, 1)$,*
(ii$_n$) *$a_n \leqslant \gamma^{(p+2)^{n-1}} a_{n-1} \leqslant \gamma^{((p+2)^n - 1)/(p+1)} a_0$ and*

$$b_n \leqslant (\gamma^{(p+2)^{n-1}})^{p+1} b_{n-1} \leqslant \gamma^{(p+2)^n-1} b_0 \quad \textit{for } n \geqslant 1,$$

(iii$_n$)

$$f(a_n)g(a_n,b_n) \leqslant \gamma^{(p+2)^n} \frac{f(a_0)g(a_0,b_0)}{\gamma} = \frac{\gamma^{(p+2)^n}}{f(a_0)}, \quad n \geqslant 0.$$

Proof. Notice that (i) is trivial. Next we prove (ii$_n$) following an inductive procedure. So, $a_1 \leqslant \gamma a_0$ and $b_1 = b_0 f(a_0)^{p+2} g(a_0,b_0)^{p+1} \leqslant \gamma^{p+1} b_0$ if and only if $f(a_0) \geqslant 1$, and by Lemma 2.1 the result holds. If we suppose that (ii$_n$) is true, then

$$\begin{aligned} a_{n+1} &= a_n f(a_n)^2 g(a_n,b_n), \\ &\leqslant \gamma^{(p+2)^{n-1}} a_{n-1} f(\gamma^{(p+2)^{n-1}} a_{n-1})^2 g(\gamma^{(p+2)^{n-1}} a_{n-1}, (\gamma^{(p+2)^{n-1}})^{p+1} b_{n-1}) \\ &\leqslant \gamma^{(p+2)^{n-1}} a_{n-1} f(a_{n-1})^2 (\gamma^{(p+2)^{n-1}})^{p+1} g(a_{n-1},b_{n-1}) = \gamma^{(p+2)^n} a_n. \end{aligned}$$

In addition, we have

$$b_{n+1} = b_n f(a_n)^{p+2} g(a_n,b_n)^{p+1} < b_n [f(a_n)^2 g(a_n,b_n)]^{p+1} \leqslant \left(\frac{a_{n+1}}{a_n}\right)^{p+1} b_n$$

and this is true since $f(a_n) > 1$. Now, as $a_{n+1}/a_n \leqslant \gamma^{(p+2)^n}$, (ii$_n$) also holds. Moreover,

$$a_{n+1} \leqslant \gamma^{(p+2)^n} a_n \leqslant \gamma^{(p+2)^n} \gamma^{(p+2)^{n-1}} a_{n-1} \leqslant \cdots \leqslant \gamma^{((p+2)^{n+1}-1)/(p+1)} a_0,$$

$$\begin{aligned} b_{n+1} &\leqslant (\gamma^{(p+2)^n})^{p+1} b_n \leqslant (\gamma^{(p+2)^n})^{p+1} (\gamma^{(p+2)^{n-1}})^{p+1} b_{n-1} \\ &\leqslant \cdots \leqslant \gamma^{(p+2)^{n+1}-1} b_0 = \frac{1}{\gamma} \gamma^{(p+2)^{n+1}} b_0. \end{aligned}$$

Finally, we observe that

$$\begin{aligned} f(a_n)g(a_n,b_n) &\leqslant f(\gamma^{((p+2)^n-1)/(p+1)} a_0) g(\gamma^{((p+2)^n-1)/(p+1)} a_0, \gamma^{(p+2)^n-1} b_0) \\ &\leqslant \gamma^{(p+2)^n} \frac{f(a_0)g(a_0,b_0)}{\gamma} = \frac{\gamma^{(p+2)^n}}{f(a_0)} = \gamma^{(p+2)^n} \Delta, \end{aligned}$$

where $\Delta = 1/f(a_0) < 1$, and the proof is complete. $\square$

3. A first result of semilocal convergence

In this section we study the sequences $\{a_n\}$ and $\{b_n\}$, defined above and prove the convergence of the sequence $\{x_n\}$ given by (1).

Notice that

$$\|L_F(x_0)\| \leqslant M \|\Gamma_0\| \|\Gamma_0 F(x_0)\| \leqslant a_0, \quad K \|\Gamma_0\| \|\Gamma_0 F(x_0)\|^{p+1} \leqslant b_0,$$

$$\|y_0 - x_0\| \leqslant \|\Gamma_0 F(x_0)\| \leqslant \eta < R\eta$$

and

$$\|x_1 - x_0\| \leqslant \left(1 + \frac{a_0}{2}\right) \|\Gamma_0 F(x_0)\| < \left(1 + \frac{a_0}{2}\right) \frac{1}{1-\gamma\Delta} \eta = R\eta.$$

since $\gamma\Delta < \Delta < 1$, then $y_0, x_1 \in B(x_0, R\eta) = \{x \in X \mid \|x - x_0\| < R\eta\}$.

In these conditions we prove, for $n \geqslant 1$, the following statements:

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leqslant f(a_{n-1})\|\Gamma_{n-1}\|$,

(II_n) $\|\Gamma_n F(x_n)\| \leqslant f(a_{n-1})g(a_{n-1}, b_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$,

(III_n) $\|L_F(x_n)\| \leqslant M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leqslant a_n$,

(IV_n) $K\|\Gamma_n\|\|\Gamma_n F(x_n)\|^{p+1} \leqslant b_n$,

(V_n) $\|x_{n+1} - x_n\| \leqslant (1 + a_n/2)\|\Gamma_n F(x_n)\|$,

(VI_n) $y_n, x_{n+1} \in B(x_0, R\eta)$.

Assuming $(1 + a_0/2)a_0 < 1$ and $x_1 \in \Omega$, we have

$$\begin{aligned}\|I - \Gamma_0 F'(x_1)\| &\leqslant \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \\ &\leqslant \|\Gamma_0\| \sup_{t\in(0,1)} \|F''(x_0 + t(x_1 - x_0))\|\|x_1 - x_0\| \leqslant M\|\Gamma_0\|\|x_1 - x_0\| \\ &\leqslant \left(1 + \frac{a_0}{2}\right) a_0 < 1.\end{aligned}$$

Then, by the Banach lemma, Γ_1 is defined and

$$\|\Gamma_1\| \leqslant \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\|\|F'(x_0) - F'(x_1)\|} \leqslant f(a_0)\|\Gamma_0\|.$$

On the other hand, we obtain from Taylor's formula

$$\begin{aligned}F(x_{m+1}) &= F(y_m) + F'(y_m)(x_{m+1} - y_m) + \int_{y_m}^{x_{m+1}} F''(x)(x_{m+1} - x)\,\mathrm{d}x, \\ &= \int_0^1 [F''(x_m + t(y_m - x_m)) - F''(x_m)](y_m - x_m)^2(1 - t)\,\mathrm{d}t \\ &\quad + \int_0^1 F''(x_m + t(y_m - x_m))(x_{m+1} - y_m)(y_m - x_m)\,\mathrm{d}t \\ &\quad + \int_0^1 F''(y_m + t(x_{m+1} - y_m))(x_{m+1} - y_m)^2(1 - t)\,\mathrm{d}t. \end{aligned} \tag{5}$$

Then, for $m = 0$, if $y_0 \in \Omega$ we have

$$\|\Gamma_1 F(x_1)\| \leqslant \|\Gamma_1\|\,\|F(x_1)\| \leqslant f(a_0)g(a_0, b_0)\,\|\Gamma_0 F(x_0)\|$$

and (II_1) is true. To prove (III_1) and (IV_1), notice that

$$\|L_F(x_1)\| \leqslant M\|\Gamma_1\|\,\|\Gamma_1 F(x_1)\| \leqslant M f(a_0)^2\|\Gamma_0\| g(a_0, b_0)\|\Gamma_0 F(x_0)\| \leqslant a_1,$$

$$K\|\Gamma_1\|\,\|\Gamma_1 F(x_1)\|^{p+1} \leqslant K f(a_0)\|\Gamma_0\| f(a_0)^{p+2} g(a_0, b_0)^{p+1}\|\Gamma_0 F(x_0)\|^{p+1} \leqslant b_1.$$

In addition, we easily deduce that

$$\begin{aligned}\|y_1 - x_0\| &\leqslant \|y_1 - x_1\| + \|x_1 - x_0\| \leqslant \left[f(a_0)g(a_0, b_0) + \left(1 + \frac{a_0}{2}\right)\right]\eta \\ &\leqslant \left[\left(1 + \frac{a_0}{2}\right) f(a_0)g(a_0, \tilde{b}_0) + \left(1 + \frac{a_0}{2}\right)\right]\eta \leqslant \left(1 + \frac{a_0}{2}\right)[1 + \gamma\Delta]\,\eta \\ &\leqslant \left(1 + \frac{a_0}{2}\right)\frac{1}{1 - \gamma\Delta}\,\eta = R\eta\end{aligned}$$

and

$$\|x_2 - x_1\| \leqslant \left(1 + \frac{a_1}{2}\right) \|\Gamma_1 F(x_1)\|.$$

Finally,

$$\|x_2 - x_0\| \leqslant \|x_2 - x_1\| + \|x_1 - x_0\| \leqslant \left[\left(1 + \frac{a_0}{2}\right)\gamma\Delta + \left(1 + \frac{a_0}{2}\right)\right]\eta$$
$$= \left(1 + \frac{a_0}{2}\right)[1 + \gamma\Delta]\,\eta < \left(1 + \frac{a_0}{2}\right)\frac{1}{1 - \gamma\Delta}\,\eta = R\eta.$$

Then, $y_1, x_2 \in B(x_0, R\eta)$ and this proof holds by induction for all $n \in \mathbb{N}$.

Now, following an inductive procedure and assuming

$$y_n, x_{n+1} \in \Omega \quad \text{and} \quad (1 + a_n/2)a_n < 1\,,\ n \in \mathbb{N}, \tag{6}$$

the items (I_n)–(VI_n) are proved.

We must now analyse the real sequences $\{a_n\}$ and $\{b_n\}$ to study the sequence $\{x_n\}$ defined in a Banach space. To establish the convergence of $\{x_n\}$ we only have to prove that it is a Cauchy sequence and that the above assumptions (6) are true. We will now show that $(1 + a_n/2)\|\Gamma_n F(x_n)\|$ is a Cauchy sequence. We note that

$$\left(1 + \frac{a_n}{2}\right)\|\Gamma_n F(x_n)\| \leqslant \left(1 + \frac{a_0}{2}\right) f(a_{n-1})g(a_{n-1}, b_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$$
$$\leqslant \cdots \leqslant \left(1 + \frac{a_0}{2}\right)\|\Gamma_0 F(x_0)\| \prod_{k=0}^{n-1} f(a_k)g(a_k, b_k).$$

We next analyse the factor $\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)$. As a consequence of Lemma 2.4 it follows that

$$\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k) \leqslant \prod_{k=0}^{n-1}\left(\gamma^{(p+2)^k}\Delta\right) = \gamma^{((p+2)^n - 1)/(p+1)}\Delta^n.$$

So, from $\Delta < 1$, we deduce that $\prod_{k=0}^{n-1} f(a_k)g(a_k, b_k)$ converges to zero by letting $n \to \infty$.

We can now state the following result on convergence for (1).

Theorem 3.1. *In the conditions indicated for the operator F, let us assume that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y, X)$ exists at some $x_0 \in \Omega$ and (c_1)–(c_4) are satisfied. Suppose that $0 < a_0 < \frac{1}{2}$ and $b_0 < ((p+1)(p+2)/8)(1 - 2a_0)(a_0 + 2)(4 - 2a_0 - a_0^2)$. Then, if $\overline{B(x_0, R\eta)} = \{x \in X;\, \|x - x_0\| \leqslant R\eta\} \subseteq \Omega$, the sequence $\{x_n\}$ defined in (1) and starting at x_0 has, at least, R-order $(p+2)$ and converges to a solution x^* of the equation $F(x) = 0$. In that case, the solution x^* and the iterates x_n, y_n belong to $\overline{B(x_0, R\eta)}$, and x^* is the only solution of $F(x) = 0$ in $B(x_0, 2/M\beta - R\eta) \cap \Omega$.*

Furthermore, we can give the following error estimates:

$$\|x^* - x_n\| \leqslant \left(1 + \frac{a_0}{2}\gamma^{((p+2)^n - 1)/(p+1)}\right)\gamma^{((p+2)^n - 1)/(p+1)}\frac{\Delta^n}{1 - \gamma^{(p+2)^n}\Delta}\,\eta. \tag{7}$$

Proof. Let us now prove (6). As $a_0 \in (0, \frac{1}{2})$ then $0 < 2 - 2a_0 - a_0^2$ and therefore

$$\left(1 + \frac{a_n}{2}\right) a_n < \left(1 + \frac{a_0}{2}\right) a_0 < 1.$$

In addition, as $y_n, x_n \in B(x_0, R\eta)$ for all $n \in \mathbb{N}$, then $y_n, x_n \in \Omega$, $n \in \mathbb{N}$.

So, (6) follows.

Now, we prove that $\{x_n\}$ is a Cauchy sequence. To do this, we consider $n, m \geqslant 1$:

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&\leqslant \left(1 + \frac{a_n}{2}\right) \eta \left[\prod_{j=0}^{n+m-2} f(a_j) g(a_j, b_j) + \cdots + \prod_{j=0}^{n-1} f(a_j) g(a_j, b_j)\right] \\
&\leqslant \left(1 + \frac{a_n}{2}\right) \left[\gamma^{\frac{(p+2)^{n+m-1}-1}{p+1}} \Delta^{n+m-1} + \cdots + \gamma^{\frac{(p+2)^n-1}{p+1}} \Delta^n\right] \eta \\
&\leqslant \left(1 + \frac{a_0}{2} \gamma^{\frac{(p+2)^n-1}{p+1}}\right) \gamma^{\frac{(p+2)^n-1}{p+1}} \Delta^n \left[\gamma^{\frac{(p+2)^n[(p+2)^{m-1}-1]}{p+1}} \Delta^{m-1}\right. \\
&\quad \left. + \gamma^{\frac{(p+2)^n[(p+2)^{m-2}-1]}{p+1}} \Delta^{m-2} + \cdots + \gamma^{\frac{(p+2)^n[(p+2)-1]}{p+1}} \Delta + 1\right]
\end{aligned}$$

By the Bernouilli inequality: $(1 + x)^k > 1 + kx$, we have

$$\|x_{n+m} - x_n\| \leqslant \left(1 + \frac{a_0}{2} \gamma^{\frac{(p+2)^n-1}{p+1}}\right) \gamma^{\frac{(p+2)^n-1}{p+1}} \Delta^n \frac{1 - \gamma^{(p+2)^n m} \Delta^m}{1 - \gamma^{(p+2)^n} \Delta} \eta, \tag{8}$$

then $\{x_n\}$ is a Cauchy sequence.

Now, by letting $m \to \infty$ in (8), we obtain (7).

From (7) it follows

$$\|x^* - x_n\| \leqslant \left(1 + \frac{a_0}{2}\right) \frac{\eta}{(1 - \Delta)\gamma^{1/(p+1)}} \left(\gamma^{\frac{1}{(p+1)}}\right)^{(p+2)^n}$$

and therefore, $\{x_n\}$ has R-order $p + 2$ at least.

To prove that $F(x^*) = 0$, notice that $\|\Gamma_n F(x_n)\| \to 0$ by letting $n \to \infty$. As $\|F(x_n)\| \leqslant \|F'(x_n)\| \times \|\Gamma_n F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is a bounded sequence, we deduce $\|F(x_n)\| \to 0$ and then $F(x^*) = 0$ by the continuity of F.

Now, to show the uniqueness, suppose that $y^* \in B(x_0, 2/N\beta - R\eta) \cap \Omega$ is another solution of $F(x) = 0$. Then

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, \mathrm{d}t (y^* - x^*).$$

Using the estimate

$$\begin{aligned}\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\| \, \mathrm{d}t &\leqslant M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\| \, \mathrm{d}t \\ &\leqslant M\beta \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|) \, \mathrm{d}t \\ &< \frac{M\beta}{2}\left(R\eta + \frac{2}{M\beta} - R\eta\right) = 1,\end{aligned}$$

we have that the operator $\int_0^1 F'(x^* + t(y^* - x^*)) \, \mathrm{d}t$ has an inverse and consequently, $y^* = x^*$. □

We now give an example to illustrate the previous results. We use a function quoted as a test in several papers (see [5]).

Example 1. Let us consider $F : C[0,1] \to C[0,1]$ to be the operator defined by

$$F(x)(s) = x(s) - s + \frac{1}{2}\int_0^1 s\cos(x(t)) \, \mathrm{d}t, \tag{9}$$

where $C[0,1]$ is the space of all continuous functions defined on the interval $[0,1]$ with the sup norm $\|\cdot\| = \|\cdot\|_\infty$.

If we choose $x_0 = x_0(s) = s$, it is easy to prove that

$$F(x_0)(s) = \frac{\sin 1}{2 - \sin 1 + \cos 1}\, s,$$

$$[F'(x_0)]^{-1} z(s) = z(s) + \frac{\int_0^1 z(s) \sin s \, \mathrm{d}s}{2 - \sin 1 + \cos 1}\, s.$$

$$F''(x)(yz)(s) = -\frac{s}{2}\int_0^1 \cos x(t) . z(t) . y(t) \, \mathrm{d}t.$$

Therefore the parameters appearing in Theorem 3.1 are

$$\|\Gamma_0\| \leqslant \frac{3 - \sin 1}{2 - \sin 1 + \cos 1} = \beta = 1.2705964\ldots$$

$$\|\Gamma_0 F(x_0)\| = \frac{\sin 1}{2 - \sin 1 + \cos 1} = \eta = 0.4953234\ldots$$

$K = M = \frac{1}{2}, a_0 = 0.314678\ldots < 0.5,\ b_0 = 0.155867\ldots < \frac{(p+1)(p+2)}{8}(1 - 2a_0)(a_0 + 2)(4 - 2a_0 - a_0^2) = 2.105095\ldots$. The conditions of Theorem 3.1 are therefore met and we obtain the solutions existence domain $B(x_0, 0.655042\ldots)$ and its uniqueness, $B(x_0, 2.49309\ldots)$.

We give an upper bound C to number $10^{11}\|x^* - x_2\|$, where x_2 is the second iterate of (1). Carrying out the same decomposition as Candela and Marquina in [4] and calculating the smallest value of n so that $\|x^* - x_2\|$ is of order 10^{-11}, we get

$$\|x^* - x_2\| \leqslant \|x^* - x_4\| + \|x_4 - x_3\| + \|x_3 - x_2\|,$$

and $C = 17\,013\,500$ is obtained. For the same function and iterative method, Candela and Marquina obtained $C = 37022683.427694$ (see [4]), meaning that we have slightly improved on the result.

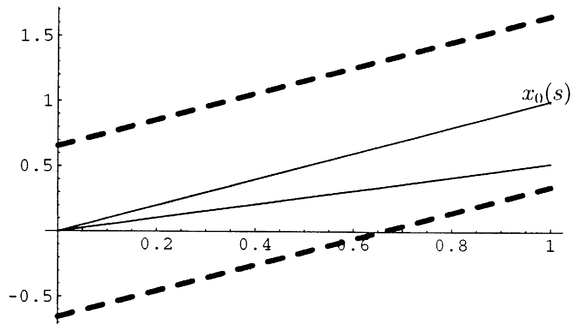

Fig. 1.

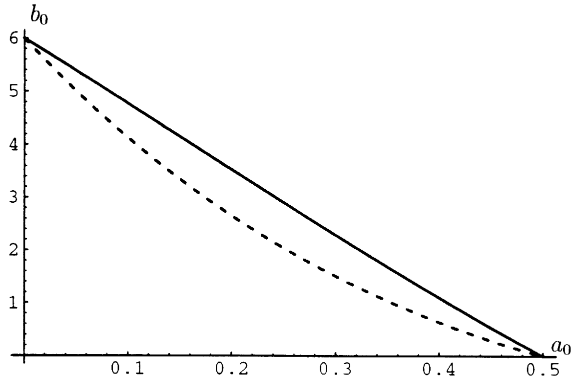

Fig. 2. Cubic decreasing regions.

Fig. 1 shows the existence domain obtained for the solution, the initial point $x_0(s)$ and the solution of (9), the line $y = ks$ with $k = 0.5224366093993515$.

To finish this convergence study of the Chebyshev method, we see that when $p = 1$ the cubic decreasing regions (see [3,4] for definition) of the Chebyshev method are represented in Fig. 2, where a_0 and b_0 are taken as coordinates. The dotted line represents the curve

$$b_0 = \frac{6(1-2a_0)(4-2a_0-a_0^2)}{(2+a_0)^2}$$

obtained by Candela and Marquina and the continuous line represents our curve

$$b_0 = \tfrac{3}{4}(1-2a_0)(a_0+2)(4-2a_0-a_0^2).$$

In consequence, our cubic decreasing region is bigger, and therefore the region of accessibility for the Chebyshev method has been increased.

4. Mild convergence conditions

Until now, the necessary conditions for the convergence of (1) have been established by assuming that the second Fréchet derivative of F is bounded in Ω. Our goal in this section is to prove the convergence of (1) assuming just that F'' is bounded only on x_0.

Considering the notation used in the previous section, let F be as in the previously indicated conditions and we suppose that conditions (c_1), (c_2), (c_4) are satisfied; we now give (c_3'): $\|F''(x_0)\| \leqslant N$.

We suppose that a solution exists in $(0, \frac{1}{2})$ of the equation

$$x = N\beta\eta + b_0 \left[\frac{2+x}{2(1-f(x)g(x,b_0))}\right]^p, \tag{10}$$

where b_0, f and g are given in (2)–(4). We will denote $\tilde{a}_0$ as this solution, $\tilde{R} = (1+\tilde{a}_0/2)1/(1-\gamma\tilde{\Delta})$ and $\tilde{\Delta} = 1/f(\tilde{a}_0)$.

We note that as $\tilde{a}_0$ is a solution to (10) it is verified that $\tilde{a}_0 = (N + K(\tilde{R}\eta)^p)\beta\eta = M\beta\eta$, where $M = N + K(\tilde{R}\eta)^p$. Defining the sequences $\{\tilde{a}_n\}$ and $\{\tilde{b}_n\}$ as above,

$$\tilde{a}_0 = M\beta\eta, \qquad \tilde{b}_0 = K\beta\eta^{p+1},$$

$$\tilde{a}_{n+1} = \tilde{a}_n f(\tilde{a}_n)^2 g(\tilde{a}_n, \tilde{b}_n) \quad \text{and} \quad \tilde{b}_{n+1} = \tilde{b}_n f(\tilde{a}_n)^{p+2} g(\tilde{a}_n, \tilde{b}_n)^{p+1},$$

we obtain the same results as seen in the Preliminaries section for these sequences.

We note that in order to apply the argument of the previous section, we need to prove that F'' is bounded in the points of the sequence $\{x_n\}$ and in the segments which join the points of sequences $\{x_n\}$ and $\{y_n\}$ (see condition (III_n) and (5)).

We will therefore prove first that $\|F''(x)\| \leqslant M$ for all $x \in B(x_0, \tilde{R}\eta)$. Let be $x \in B(x_0, \tilde{R}\eta)$, then

$$\|F''(x)\| \leqslant \|F''(x_0)\| + \|F''(x) - F''(x_0)\| \leqslant N + K\|x - x_0\|^p \leqslant N + K(\tilde{R}\eta)^p = M.$$

Once this condition is proved, and given that

$$\|L_F(x_0)\| \leqslant N\beta\eta \leqslant M\beta\eta = \tilde{a}_0,$$

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta \leqslant \tilde{R}\eta$$

and

$$\|x_1 - x_0\| \leqslant \left(1 + \frac{\tilde{a}_0}{2}\right)\|\Gamma_0 F(x_0)\| < \left(1 + \frac{\tilde{a}_0}{2}\right)\frac{1}{1-\gamma\tilde{\Delta}}\,\eta = \tilde{R}\eta,$$

that is $y_0, x_1 \in B(x_0, \tilde{R}\eta)$, the following conditions can be proved for $n \geqslant 1$ by analogy with the previous section.

(I_n) $\|\Gamma_n\| = \|F'(x_n)^{-1}\| \leqslant f(\tilde{a}_{n-1})\|\Gamma_{n-1}\|$,
(II_n) $\|\Gamma_n F(x_n)\| \leqslant f(\tilde{a}_{n-1})g(\tilde{a}_{n-1}, \tilde{b}_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$,
(III_n) $\|L_F(x_n)\| \leqslant M\|\Gamma_n\|\ \|\Gamma_n F(x_n)\| \leqslant \tilde{a}_n$,
(IV_n) $K\|\Gamma_n\|\ \|\Gamma_n F(x_n)\|^{p+1} \leqslant \tilde{b}_n$,
(V_n) $\|x_{n+1} - x_n\| \leqslant (1 + a_n/2)\|\Gamma_n F(x_n)\|$,
(VI_n) $y_n, x_{n+1} \in B(x_0, \tilde{R}\eta)$.

The following result is therefore proved.

Theorem 4.1. *Let F be as in the usual conditions. Let us assume that $\Gamma_0 = F'(x_0)^{-1} \in \mathscr{L}(Y,X)$ exists at some $x_0 \in \Omega$ and* $(c_1), (c_2), (c_3')$ *and* (c_4) *are satisfied. Let us denote $\tilde{a}_0 = M\beta\eta$ and $\tilde{b}_0 = K\beta\eta^{p+1}$ and suppose that $\tilde{b}_0 < ((p+1)(p+2)/8)(1-2\tilde{a}_0)(\tilde{a}_0+2)(4-2\tilde{a}_0-\tilde{a}_0^2)$. Then, if $\overline{B(x_0,\tilde{R}\eta)} = \{x \in X; \|x-x_0\| \leqslant \tilde{R}\eta\} \subseteq \Omega$, the sequence $\{x_n\}$ defined in* (1) *and starting at x_0 has, at least, R-order $(p+2)$ and converges to a solution x^* of the equation $F(x) = 0$. In that case, the solution x^* and the iterates x_n, y_n belong to $\overline{B(x_0,\tilde{R}\eta)}$, and x^* is the only solution of $F(x) = 0$ in $B(x_0,(2/M\beta)-\tilde{R}\eta)\cap\Omega$.*

Furthermore, we can give the following error estimates:

$$\|x^* - x_n\| \leqslant \left(1 + \frac{\tilde{a}_0}{2}\,\tilde{\gamma}^{((p+2)^n-1)/(p+1)}\right)\tilde{\gamma}^{((p+2)^n-1)/(p+1)}\,\frac{\tilde{\Delta}^n}{1-\tilde{\gamma}^{(p+2)^n}\tilde{\Delta}}\,\eta, \tag{11}$$

where $\tilde{\gamma} = \tilde{a}_1/\tilde{a}_0$.

We can illustrate this result with the following example

Example 2. Let us consider $F : C[0,1] \to C[0,1]$ with the operator defined by

$$F(x)(s) = x(s) - 1 - \frac{1}{4}\int_0^1 \frac{s}{s+t}\,x(t)^{11/5}\,\mathrm{d}t, \tag{12}$$

where $C[0,1]$ is the space of all continuous functions defined in the interval $[0,1]$ with the sup norm $\|\cdot\| = \|\cdot\|_\infty$. It is easy to prove

$$[F'(x)(y)](s) = y(s) - \frac{11}{20}\int_0^1 \frac{s}{s+t}\,x(t)^{6/5}y(t)\,\mathrm{d}t$$

and

$$[F''(x)(yz)](s) = -\frac{33}{50}\int_0^1 \frac{s}{s+t}\,x(t)^{1/5}z(t)y(t)\,\mathrm{d}t,$$

and we obtain that

$$\|F''(x)\| \leqslant \frac{33}{50}\|x\|^{1/5}\log 2. \tag{13}$$

We can see that (13) depends on the norm of x, and it cannot therefore be bounded in Ω; consequently, condition (c_3) is not satisfied and therefore Theorem 3.1. cannot be applied. This problem disappears if Theorem 4.1 is applied, which only requires that the second derivative be bounded in the initial point x_0. If we choose $x_0 = x_0(s) = 1$, we have

$$F(x_0)(s) = -\frac{s}{4}\log\frac{s+1}{s},$$

$$\|F(x_0)(s)\| = \frac{\log 2}{4}.$$

The existence of Γ_0 will now be proved and we will calculate the parameters appearing in Theorem 4.1:

$$\|[I - F'(x_0)]\| = \max_{s\in[0,1]} |y(s) - F'(x_0)y(s)| = \tfrac{11}{20}\log 2 = 0.381231\ldots < 1.$$

Table 1
Weights and nodes for the Gauss–Legendre formula for $m = 8$

j	1	2	3	4	5	6	7	8
t_j	0.01985507	0.10166676	0.237233795	0.40828268	0.59171732	0.762766205	0.89833324	0.98014493
β_j	0.10122854	0.22381034	0.31370665	0.36268378	0.36268378	0.31370665	0.22381034	0.10122854

Table 2
Solutions of (12)

i	1	2	3	4	5	6	7	8
x_i	1.02415468657	1.07954327274	1.13201750646	1.17280585507	1.20186439201	1.22141388151	1.23360029059	1.23991266325

The Banach lemma gives the existence of Γ_0 and we obtain

$$\|\Gamma_0\| \leqslant \frac{20}{20 - 11\log 2} = \beta = 1.61611\ldots$$

$$\|\Gamma_0 F(x_0)\| = \frac{5\log 2}{20 - 11\log 2} = \eta = 0.280051\ldots$$

$K = N = 0.457477\ldots, a_0 = 0.383653\ldots < 0.5,\ b_0 = 0.160518\ldots < ((p+1)(p+2)/8)(1-2a_0)(a_0 + 2)(4 - 2a_0 - a_0^2) = 0.5647661\ldots$. The conditions of Theorem 4.1 are therefore met and we obtain the solutions existence domain in $B(x_0, 0.451426\ldots)$ and uniqueness in $B(x_0, 1.00849\ldots)$.

Finally, Eq. (12) is discretized to replace it by a finite dimension problem. The integral appearing in (12) is approximated by a numerical integration formula, using the Gauss–Legendre formula

$$\int_0^1 f(t)\,\mathrm{d}t \approx \frac{1}{2}\sum_{j=1}^{m} \beta_j f(t_j)$$

for $m = 8$ where t_j and β_j are the known nodes and weights which appear in Table 1.

Denoting the approximations of $x(t_i), i = 1,\ldots,8$, as x_i we reach the following system of non-linear equations:

$$x_i = 1 + \frac{t_i}{8}\sum_{j=1}^{8} \beta_j \frac{x_j^{\frac{11}{5}}}{t_i + t_j}, \quad i = 1,\ldots,8.$$

If we then use a_{ij} to mean $\frac{1}{8}t_i\beta_j/(t_i + t_j)$, we can write the above system in the form

$$x_i = 1 + \sum_{j=1}^{8} a_{ij} x_j^{11/5}, \quad i = 1,\ldots,8. \tag{14}$$

The solution which appears in Table 2 is obtained by using the Mathematica programme

Interpoling the function which passes through points (t_i, x_i) $i = 1,\ldots,8$, and knowing that $x(0) = 1$, we obtain the graph approximating to the solution (Fig. 3(a)). Fig. 3(b) shows that the solution obtained is found in the solution existence domain located for the non-linear integral equation considered.

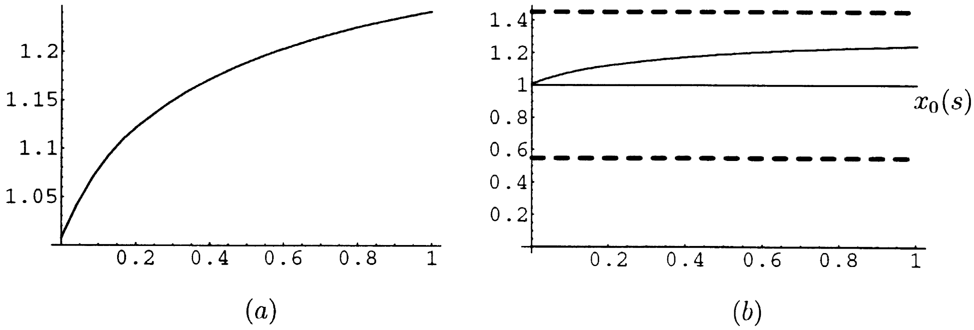

Fig. 3.

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